Language Robot Controller Supplemental

1 Passivity Criteria

A passive system is a system that constrained in such a way that it does not inject excessive energy or instability into the interaction [**?**]. More formally a system is passive with respect to an input output pair $(u(t), y(t))$ if and only if there exists a positive definite storage function V over the system such that:

$$V(t) - V(0) \leq \int_0^t u(t)^T \cdot y(t) dt \quad \forall t > 0 \tag{1}$$

Theorem 1: Consider the system following a trajectory T with dynamics governed by:

$$M_0 \dot{v}_{\text{ref}}(t) = p_t^2 (F_{\text{ext}} + F_{\text{virtual}}) - (p_t D_0 - \tfrac{\dot{p}_t}{p_t} M_0) v_{\text{ref}} \tag{2}$$

$$\text{where } F_{\text{virtual}} = \underbrace{K(x_d - x)}_{F_{\text{guide}}} + \underbrace{\|F_{\text{propell}}\| \mathbf{b}}_{F_{\text{propell}}}, \tag{3}$$

$$\text{and } \mathbf{b} = \lim_{d' \to d^+} \frac{T(d') - x_d}{\|T(d') - x_d\|}, \tag{4}$$

If F_{propell} is a positive definite function M_0, D_0 and K_0 are positive definite matrices, K_0 is orthogonal, and

$$\exists\, p_{\text{min}} > 0 \text{ such that } p_{\text{min}} < p_t(t) \quad \forall t > 0$$

then for any linear trajectory T, the system is passive with respect to the the force-velocity $(F_{\text{ext}}, v_{\text{ref}})$ input output pair.

Proof: Differentiating (1), one sees it is sufficient to show that there exists positive semi-definite storage function such that:

$$\dot{V} \leq u(t)^T \cdot y(t) \quad \forall t > 0 \tag{5}$$

We dedicate the rest of the section to constructing such a function. We first proceed with some notation. Following the paper, T, denotes the oriented line drawn out by our trajectory in state space and $\mathbf{b}$ denote the unit vector parallel to T. Since the path is linear there exists a fixed vector $\mathbf{N}$ parallel to the trajectory such that $\mathbf{b}$ lies parallel to $\mathbf{N}$ for all points on the path.

For any vector x, let $x_{||}$ denote the component parallel to $\mathbf{N}$ and let $x_\perp$ denote the component perpendicular to N. Since $x_{||}$ is the projection of x onto the span of $\mathbf{N}$, it is the closest point on T. Therefore :

$$x - x_d = x - x_{||} = x_\perp \tag{6}$$

Substituting (3) into (2) yields:

$$M\dot{v}_{ref} = {p_t}^2 K_0(x_d - x) - (p_t D_0 - M_0 \frac{\dot{p}_t}{p_t}) v_{\text{ref}} - {p_t}^2 F_{\text{propell}}(x_d)\mathbf{b} + p_t^2 F_{\text{ext}} \tag{7}$$

Substituting (6) into (7) yields:

$$M\dot{v}_{ref} = -{p_t}^2 K_0 x_\perp - (p_t D_0 - M_0 \frac{\dot{p}_t}{p_t}) v_{\text{ref}} + {p_t}^2 F_{\text{propell}}(x_d)\mathbf{b} + p_t^2 F_{ext} \tag{8}$$

We define our storage function V as follows :

$$V(x, v_{\text{ref}}, p) = \tfrac{1}{2}\, x_\perp^T K_0 x_\perp + \phi(x_{||}) + \tfrac{1}{2} p^{-2} {v_{\text{ref}}}^T M_0 v_{\text{ref}} \tag{9}$$

where $\phi(x_{||})$ is the line integral of the propelling force along the trajectory to the origin, defined as $\phi(x_{||}) = \int_{x_{||}}^{0} F_{\text{propell}}(x)\mathbf{b} \cdot dT$. [1]

We claim V is positive semi definite and satisfies (5). We first show our function is positive definite. Since K_0 is positive definite:

$$\forall x_\perp \neq 0, \quad x_\perp^T K_0 x_\perp > 0 \tag{10}$$

$$x_\perp = 0 \implies x_\perp^T K_0 x_\perp = 0 \tag{11}$$

Since M_0 is positive definite and $p_t > p_{\text{min}} > 0$:

$$\forall v_{\text{ref}} \neq 0, \quad p_t^{-2} v_{\text{ref}}^T M_0 v_{\text{ref}} > 0 \tag{12}$$

$$v_{\text{ref}} = 0 \implies p_t^{-2} v_{\text{ref}}^T M_0 v_{\text{ref}} = 0 \tag{13}$$

Since $F_{\text{propell}}(x_d)\mathbf{b}$ lies parallel to the trajectory, the line integral along the trajectory is positive, and thus

$$\forall x_{||} \neq 0, \quad \phi(x_{||}) > 0 \tag{14}$$

$$x_{||} = 0 \implies \phi(x_{||}) = 0 \tag{15}$$

Thus each term is non negative and $V = 0$ if and only if $v_{\text{ref}} = 0$ and $x = x_\perp + x_{||} = 0$. Therefore V is positive definite.

We now show V satisfies (5). Taking the time derivative of V yields: [2]

$$\dot{V} = {v_{\text{ref}}}_\perp^T K_0 x_\perp - F_{\text{propell}}(x_{||})\mathbf{b} {v_{\text{ref}}}_{||}^T - \dot{p}_t p_t^{-3} v_{\text{ref}}^T M_0 v_{\text{ref}} + p_t^{-2} {v_{\text{ref}}}^T M_0 \dot{v}_{\text{ref}} \tag{16}$$

Substituting (8) for $M_0\dot{v}_{\text{ref}}$:

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[1] dT is the tangent vector to the trajectory vector T

[2] Note $\phi(x_{||}) = \int_{x_{||}}^{0} F_{\text{propell}}(x)\mathbf{b}\ dT = \int_{0}^{x_{||}} -F_{\text{propell}}(x)\mathbf{b}\ dT$. By the chain rule $\dot{\phi}(x_{||}) = \frac{\partial}{\partial x_{||}}(\int_0^{x_{||}} -F_{\text{propell}}(x)\mathbf{b}\ dT)\dot{x}_{||}^T = -F_{\text{propell}}(x)\mathbf{b} v_{\text{ref}}^T$

$$\dot{V} = {v_{\text{ref}}}_\perp^T K_0 x_\perp - F_{\text{propell}}(x_{||})\mathbf{b}{v_{\text{ref}}}_{||}^T - \dot{p}_t p_t^{-3} v_{\text{ref}}^T M_0 v_{\text{ref}}$$

$$+p_t^{-2} v_{\text{ref}}^T(-{p_t}^2 K_0 x_\perp - (p_t D_0 - M_0 \frac{\dot{p}_t}{p_t}) v_{\text{ref}} + {p_t}^2 F_{\text{propell}}(x_d)\mathbf{b} + {p_t}^2 F_{\text{ext}}) \quad (17)$$

Distributing the $p_t^{-2} v_{\text{ref}}^T$:

$$\dot{V} = {v_{\text{ref}}}_\perp^T K_0 x_\perp - F_{\text{propell}}\mathbf{b}{v_{\text{ref}}}_{||}^T - \dot{p}_t p_t^{-3} v_{\text{ref}}^T M_0 v_{\text{ref}} - v_{\text{ref}}^T K_0 x_\perp - p_t^{-1} v_{\text{ref}}^T D_0 v_{\text{ref}}$$

$$+\dot{p}_t p_t^{-3} v_{\text{ref}}^T M_0 v_{\text{ref}} + v_{\text{ref}}^T F_{\text{propell}}\mathbf{b} + v_{\text{ref}}^T F_{\text{ext}} \quad (18)$$

Rearranging terms :

$$\dot{V} = {v_{\text{ref}}}_\perp^T K_0 x_\perp - v_{\text{ref}}^T K_0 x_\perp + \dot{p}_t p_t^{-3} v_{\text{ref}}^T M_0 v_{\text{ref}} - \dot{p}_t p_t^{-3} {v_{\text{ref}}}^T M_0 v_{\text{ref}} +$$

$$-F_{\text{propell}}\mathbf{b}{v_{\text{ref}}}_{||}^T + v_{\text{ref}}^T F_{\text{propell}}\mathbf{b} - p_t^{-1} v_{\text{ref}}^T D_0 v_{\text{ref}} + v_{\text{ref}}^T F_{\text{ext}} \quad (19)$$

Cancelling terms and rewriting $v_{\text{ref}} = v_{\text{ref}||} + v_{\text{ref}\perp}$:

$$\dot{V} = {v_{\text{ref}}}_\perp^T K_0 x_\perp - (v_{\text{ref}||} + v_{\text{ref}\perp})^T K_0 x_\perp - F_{\text{propell}}\mathbf{b}{v_{\text{ref}}}_{||}^T +$$

$$(v_{\text{ref}||} + v_{\text{ref}\perp})^T F_{\text{propell}}\mathbf{b} - p_t^{-1} v_{\text{ref}}^T D_0 v_{\text{ref}} + v_{\text{ref}}^T F_{\text{ext}} \quad (20)$$

Since K_0 an orthogonal matrix then $v_{\text{ref}||} \cdot x_\perp = 0$ implies ${v_{\text{ref}}}_{||}^T K_0 x_\perp = 0$. Since $\mathbf{b}$ is parallel to $\mathbf{N}$, then $\mathbf{N} \cdot v_{\text{ref}\perp} =$ implies $\mathbf{b} \cdot v_{\text{ref}\perp} = 0$. Therefore we may expand and simplify to yield:

$$\dot{V} = {v_{\text{ref}}}_\perp^T K_0 x_\perp - {v_{\text{ref}}}_\perp^T K_0 x_\perp - F_{\text{propell}}\mathbf{b}{v_{\text{ref}}}_{||}^T + F_{\text{propell}}\mathbf{b}{v_{\text{ref}}}_{||}^T - p_t^{-1} v_{\text{ref}}^T D_0 v_{\text{ref}} + v_{\text{ref}}^T F_{\text{ext}} \quad (21)$$

Cancelling terms:

$$\dot{V} = -p_t^{-1} v_{\text{ref}}^T D_0 v_{\text{ref}} + v_{\text{ref}}^T F_{\text{ext}} \quad (22)$$

Since D is positive definite and $p \geq p_{\text{min}} \geq 0$:

$$-p^{-1} v_{\text{ref}}^T D_0 v_{\text{ref}} \leq 0 \quad (23)$$

Therefore

$$\dot{V} = -p^{-1} v_{\text{ref}}^T D_0 v_{\text{ref}} + v_{\text{ref}}^T F_{\text{ext}} \leq v_{\text{ref}}^T F_{\text{ext}}. \quad (24)$$

Thus for any linear trajectory our system is passive with respect to the force-velocity, $(F_{\text{ext}}, v_{\text{ref}}^T)$, input output pair. Furthermore, reasonably well-behaved trajectories can be effectively approximated as a series of linear trajectories, and as more lines are added, the approximation becomes arbitrarily faithful. This robust heuristic suggests stability even for more complex nonlinear paths, demonstrating the practicality and reliability of our language controller.

References

[1] J. Wyatt, L. Chua, J. Gannett, I. Goknar, and D. Green, "Energy concepts in the state-space theory of nonlinear n-ports: Part i-passivity," *IEEE transactions on Circuits and Systems*, vol. 28, no. 1, pp. 48–61, 1981.